Generating Recursive Formulas for the Number of Spanning Trees in Cyclic Snakes Networks.

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Abstract

Calculating the number of spanning trees of a graph G by the determinant of Laplacian matrix is tedious and impractical. In this paper, we propose the combinatorial method to facilitate the calculation of the number of spanning trees for some graphs. In particular, we derive the explicit formulas for the triangular snake (Δ_k *-snake*), double triangular snake ($2\Delta_k$ *-snake*) and the total graph of path P_n ($T(P_n)$). Finally, we derive the explicit formulas for the subdivision of Δ_k *-snake*, $2\Delta_k$ *-snake* and $T(P_n)$.

Keywords: spanning trees, enumeration, cyclic snakes

1. Introduction:

We deal with simple and finite undirected graphs $G = (V, E)$, where V is the vertex set and E is the edge set. For a graph G, a spanning tree in G is a tree which has the same vertex set of G. The number of spanning trees in G, also called, the complexity of the graph, denoted by $\tau(G)$.

The number of spanning trees $\tau(G)$ in graphs (networks) is an important invariant. The evaluation of this number and analyzing its behavior is not only interesting from a mathematical (computational) perspective, but also, it is an important measure of reliability of a network and designing electrical circuits. Some computationally hard problems such as the travelling salesman problem can be solved approximately by using spanning trees. There two methods for counting the number of spanning trees in graphs, algebraic and combinatorial methods.

First, Algebraic graph theory is a branch of mathematics that studies graphs by using algebraic properties of associated matrices.

In 1847, a classical result of Kirchhoff [1] can be used to determine the number of spanning trees for $G = (V, E)$. Let $V = \{v_1, v_2, \ldots, v_n\}$, then the Kirchhoff matrix H defined as $n \times n$ characteristic matrix $H = D\text{-}A$, where D is the diagonal matrix of the degrees of G and A is the adjacency matrix of G, $H = [a_{ij}]$ defined as follows:

$$H = [a_{ij}] = \begin{cases} deg(v_i) & if \ \ i = j \\ -1 \quad if \ \ v_i v_j \in E(G) \\ 0 \quad if \ \ v_i v_j \notin E(G) \end{cases}$$

All of co-factors of H are equal to $t(G)$.

There are other methods for calculating $t(G)$. Let $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_p$ denote the eigenvalues of H matrix of a p point graph. Then it is easily shown that $\mu_p = 0$. In 1974, Kelmans and Chelnokov [2] shown that, $t(G) = \frac{1}{p}\prod_{k=1}^{p-1}\mu_k$. The formula for the number of spanning trees in a *d-regular* graph G can be expressed as $t(G) = \frac{1}{p}\prod_{k=1}^{p-1}(d - \mu_k)$ where $\lambda_0 = \lambda_1, \lambda_2, \ldots, \lambda_{p-1}$ are the eigenvalues of the corresponding adjacency matrix of the graph. However, for a few special families of graphs there exist simple formulas that make it much easier to calculate and determine the number of corresponding spanning trees especially when these numbers are very large. One of the first such results is due to Cayley [3] who showed that complete graph on n vertices, K_n has n^{n-2} spanning trees that he showed $\tau(K_n) = n^{n-2}, \ n \geq 2$. In 2003, Clark proved that $\tau(K_{p,q}) = p^{q-1}q^{p-1}, \ \ p, q \geq 1$, where $K_{p,q}$ is the complete bipartite graph with bipartite sets containing p and q vertices, respectively. Another result is due to Guy [5] who derived a formula for the wheel on n+1vertices, W_{n+1}, which is formed from a cycle C_n on n vertices by adding a vertex adjacent to every vertex of C_n. In particular, he showed that $t(W_{n+1}) = (\frac{3+\sqrt{5}}{2})^n + (\frac{3-\sqrt{5}}{2})^n - 2$ for $n \geq 3$. Sedlacek [6] also later derived a formula for the number of spanning trees in a Mobius ladder. The Mobius ladder M_n is formed from cycle C_{2n} on $2n$ vertices labeled

$v_1, v_2, \ldots, v_{2n}$ by adding edge $v_i v_{i+n}$ for every vertex v_i where $n \geq 2$. The number of spanning trees in M_n is given by $t(M_n) = \frac{n}{2}[(2+\sqrt{3})^n + (2-\sqrt{3})^n + 2]$ for $n \geq 2$. Boesch and Bogdanowicz [7] introduced another class of graphs for which an explicit formula has been derived is based on a prism. Let the vertices of two disjoint and length cycles be labeled $v_1, v_2, \ldots, v_n$ in one cycle and $w_1, w_2, \ldots w_n$ in the other. The prism R_n is defined as the graph obtained by adding to these two cycles all edges of the form v_i, w_i. The number of spanning trees in R_n is given by the following formula $\frac{n}{2}[(2+\sqrt{3})^n + (2-\sqrt{3})^n + 2]$.

Second, the basic combinatorial idea, Feussner's recursive formula [8], for counting $\tau(G)$ in a graph G is quite intuitive. The combinatorial method was used because of, for a large graph, evaluating the relevant determinant is computationally intractable. Wherefore, many works derive formulas to calculate the complexity for some classes of graphs. Bogdanowicz [9] derive the explicit formula $\tau(F_n)$; the number of spanning trees in F_n. Modabish and El Marraki investigated the number of spanning trees in the star flower planar graph [10].

In this paper, we consider the combinatorial method for finding the number of spanning trees in Cyclic Snakes Networks. In this paper, we derive the explicit formulas for the triangular snake (Δ_k *-snake*), double triangular snake ($2\Delta_k$ *-snake*) and the total graph of path P_n ($T(P_n)$). Finally, we derive the explicit formulas for the subdivision of Δ_k *-snake*, $2\Delta_k$ *-snake* .

2. Applications for the Number of spanning trees.

The number of spanning trees in a graph (network) is an important, well studied quantity [11]. As well as being of combinatorial interest, several application uses, mentioned in the following, are adduced in [12].

1. Kirchhoff's laws, well known as Matrix Tree Theorem, provide an effective method for designing electrical circuits, which are enormously useful in the analysis and synthesis of networks.

2-Suppose we are given a network of communication lines, which can break. The probability of a single line breaking is $1-p$. It is necessary to estimate the reliability of such a network. If the reliability is the probability of connectedness of the network, P, then

$$P = \sum_{k=n-1}^{m} A_k p^k (1-p)^{m-k}$$

where n is the number of vertices, m the number of edges, of the graph; A_k is the number of connected subgraphs with n vertices and k edges. It is clear that, if the reliability of each line is small, then

$$P \approx A_{n-1} p^{n-1} (1-p)^{m-n+1}$$

where A_{n-1} is the number of spanning trees of the graph.

Thus, with low reliability of each of the line, the network's reliability is determined, basically, by the number of spanning trees of the network.

3. In building a maser, one must investigate the possible particle transitions. For this, one constructs a graph in which the vertices correspond to energy levels and edges to possible particle transitions. Then for the analysis of the maser's energetics, it turns out to be very useful to know the number of spanning trees in the corresponding graph.

3. Preliminary Notes

The combinatorial method involves the operation of contraction of an edge. An edge e of a graph G is said to be contracted if it is deleted and its ends are identified. The resulting graph is denoted by $G.e$. Also we denote by $G - e$ the graph obtained from G by deleting the edge e.

Theorem 3.1 [13] Let G be a planar graph (multiple edges are allowed in here).
Then for any edge e

$$\tau(G) = \tau(G - e) + \tau(G.e)$$

Remark 1: If G' is obtained from G by removing all the pendant edges of G, then $\tau(G') = \tau(G)$.

Remark 2: If G' is obtained from G by removing all the loops of G, then $\tau(G') = \tau(G)$.

Remark 3: If G' is obtained from G by removing one or more than one multiple edges of G, then $\tau(G') < \tau(G)$.

Definition [14]:

A triangular snake (or Δ_k *-snake*) is a connected graph in which all blocks are triangles and the block-cut-point graph is a path.

Definition:

A double triangular snake is a graph formed by two triangular snake having a common path. The double triangular snake is denoted by $2\Delta_k$ - *snake*.

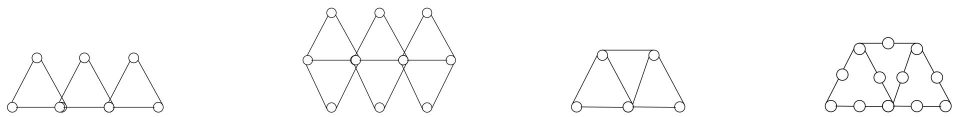
Definition [15]:

The total graph of a graph *G* is the graph whose vertex set is $V(G) \cup E(G)$ and two vertices are adjacent whenever they are either adjacent or incident in *G*. The total graph of *G* denoted by *T*(*G*).

Definition:

The subdivision of a graph *G* means the graph obtained by subdividing every edge of *G* exactly once and denoted by *S*(*G*).

Illustration:



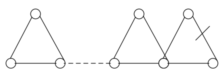
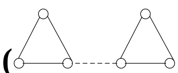
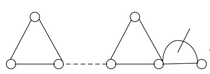
Δ_k *-snake* $\quad$ $2\Delta_k$ *-snake* $\quad$ T(P_3) $\quad$ *S*(*T*(P_3))

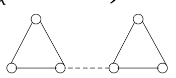
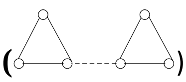
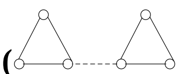
4. The Main Results:

Theorem1: The number of spanning trees of the triangular snake graph Δ_k *-snake* satisfies the following recursive relation:

$\tau(\Delta_k\text{-}snake) = 3\tau(\Delta_{k-1}\text{-}snake)$

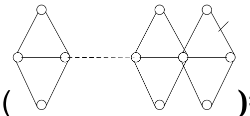
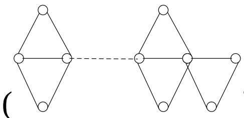
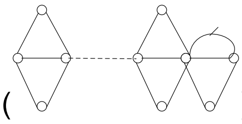
Proof:

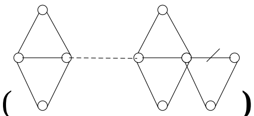
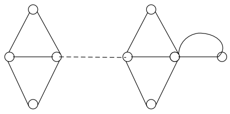
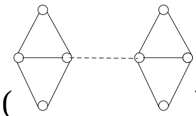
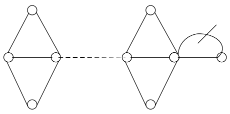
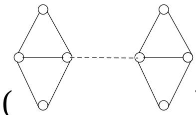
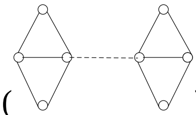
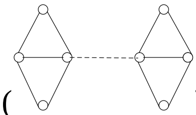
$\tau(\Delta_k\text{-}snake) = \tau($  $) = \tau($  $) + \tau($  $) =$

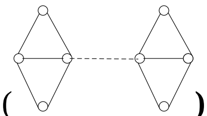
$2\tau($  $) + \tau($  $) = 3\tau($  $) = 3\tau(\Delta_{k-1}\text{-}snake)$ ■

Theorem2: The number of spanning trees of the double triangular snake graph Δ_k -*snake* satisfies the following recursive relation: $\tau($ $2\Delta_{\mathrm{k}}$-*snake* $)= 8\tau($ $2\Delta_{\mathrm{k-1}}$-*snake* $)$

Proof:

$\tau($ $2\Delta_{\mathrm{k}}$-*snake* $) = \tau($  $)= \tau($  $) + \tau($  $)$

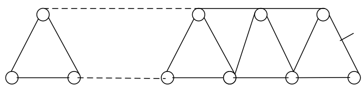
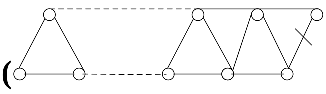
$= 2\tau($  $) + \tau($  $) = 2\tau($  $) + 3\tau($  $) = 2\tau($  $)+3\tau($  $)+3\tau($  $) =$

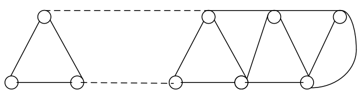
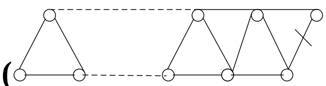
$8\tau($  $) = 8\tau($ $2\Delta_{\mathrm{k-1}}$-*snake* $)$ ■

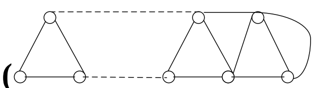
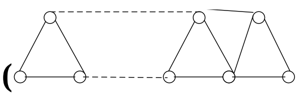
Theorem3: The number of spanning trees of the total graph of path P_n satisfies the following recursive relation:

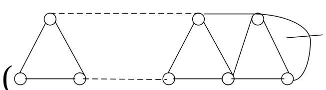
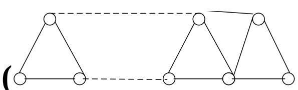
$$\tau(T(P_n)) = 7\tau(T(P_{n-1})) - \tau(T(P_{n-2}))$$

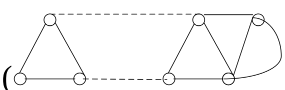
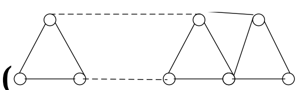
Proof:

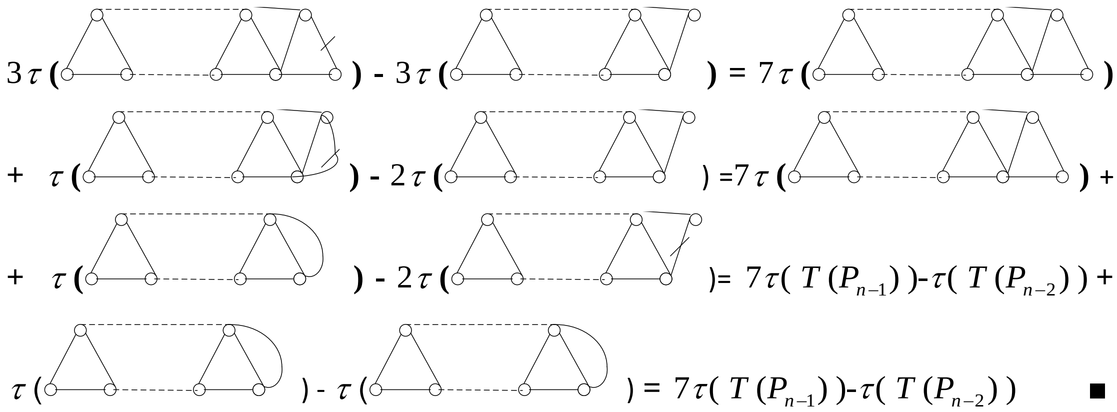
$\tau(T(P_n)) = \tau($  $) = \tau($  $) +$

$\tau($  $) = 2\tau($  $) +$

$\tau($  $) = 2\tau($  $)$

$+3\tau($  $) = 5\tau($  $) +$

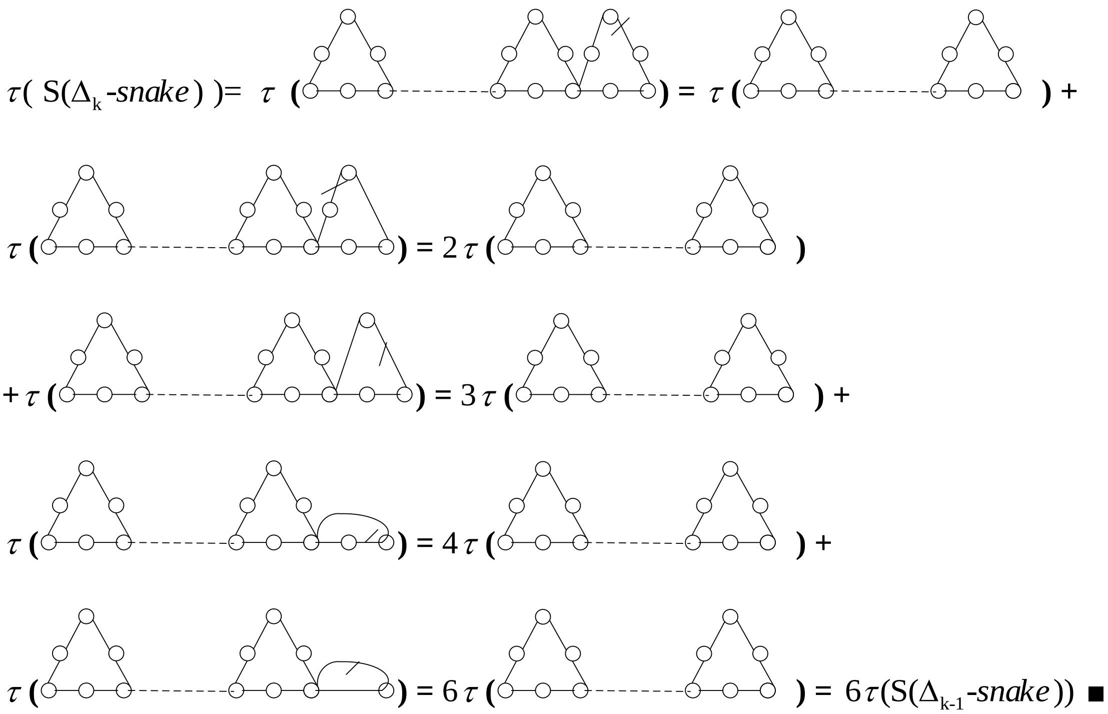
$3\tau($  $) = 5\tau($  $) +$



Theorem4: The number of spanning trees of the subdivision of triangular snake graph Δ_k -*snake* satisfies the following recursive relation:

$\tau($ S(Δ_{k}-*snake*))= 6$\tau($ S($\Delta_{\text{k-1}}$-*snake*))

Proof:

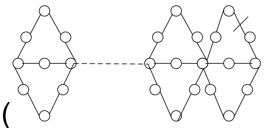
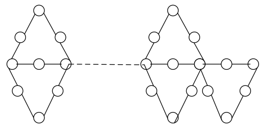


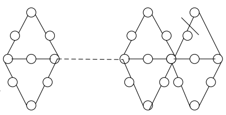
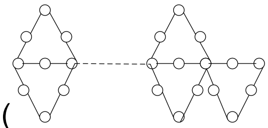
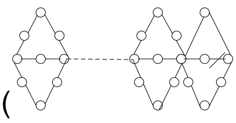
Corollary 5 : The number of spanning trees of the subdivision of triangular snake graph Δ_k -*snake* is equal to : 6^k where k is the number of blocks of Δ_k -*snake***.**

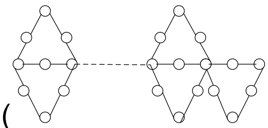
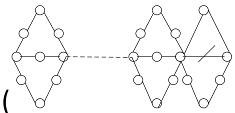
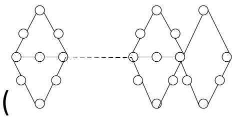
Theorem 6: The number of spanning trees of the subdivision of double triangular snake graph Δ_k -*snake* satisfies the following recursive relation:

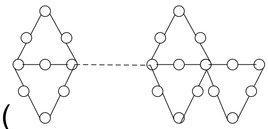
$\tau(\text{ S}(\Delta_k\text{-}snake)\text{ })= 32\,\tau(\text{ S}(\Delta_{k-1}\text{-}snake)\text{ })$

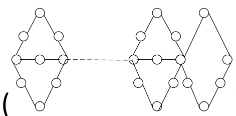
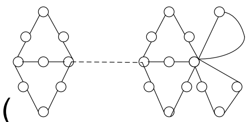
Proof:

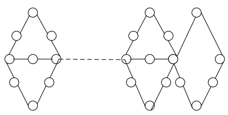
$\tau(\text{S}(\Delta_k\text{-}snake)) = \tau\,($  **)** $= \tau$ **(** )+

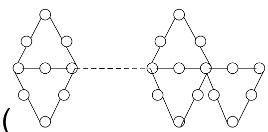
$\tau\,($ )= $2\,\tau\,($  **)** $+\,\tau\,($ ) =

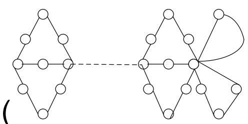
$2\,\tau\,($  **)** $+\,\tau\,($ )+ $\tau\,($ )=

$2\,\tau\,($  **)** +

$2\,\tau\,($ ) + $\tau\,($ ) **------------------------------------- (1)**

Since $\tau\,($ ) = $6\tau(\text{ S}(\Delta_{k-1}\text{-}snake)\text{ })$ **--------------------------------------(2)**

Since $\tau\,($  **)** = $6\tau(\text{ S}(\Delta_{k-1}\text{-}snake)\text{ })$ **____________________________(3)**

Since $\tau\,($ ) = $8\tau(\text{ S}(\Delta_{k-1}\text{-}snake)\text{ })$**____________________________(4)**

From 1, 2, 3 and 4 we have $\tau(\text{ S}(\Delta_k\text{-}snake)\text{ })= 32\,\tau(\text{ S}(\Delta_{k-1}\text{-}snake)\text{ })$. ■

Corollary 7 : The number of spanning trees of the subdivision of double triangular snake graph Δ_k *-snake* is equal to : $(32)^k$ where k is the number of blocks of Δ_k -*snake***.**

5. Conclusion

The number of spanning trees of a graph G is the total number of distinct spanning subgraphs of G that are trees (tree that visiting all the vertices of the graph G). Calculating the number of spanning trees of a graph G by the determinant of Laplacian matrix is tedious and impractical. In this paper, we proposed the combinatorial method to facilitate the calculation of the number of spanning trees for some graphs. In particular, we derived the explicit formulas for the triangular snake (Δ_k *-snake*), double triangular snake ($2\Delta_k$ *-snake*) and the total graph of path P_n ($T(P_n)$). Finally, we derived the explicit formulas for the subdivision of Δ_k *-snake*, $2\Delta_k$ *-snake* .

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